Average site perimeter of directed animals on the two-dimensional lattices

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Abstract

We introduce new combinatorial (bijective) methods that enable us to compute the average value of three parameters of directed animals of a given area, including the site perimeter. Our results cover directed animals of any one-line source on the square lattice and its bounded variants, and we give counterparts for most of them in the triangular lattices. We thus prove conjectures by Conway and Le Borgne. The techniques used are based on Viennot's correspondence between directed animals and heaps of pieces (or elements of a partially commutative monoid).

1 Introduction

Let Γ be an oriented graph and S a nonempty finite set of vertices of Γ. A directed animal of source S on Γ is a finite set of vertices A that contains S and such that for every vertex v of A, there exists a vertex s of S and a path from s to v going only through vertices of A. The vertices of a directed animal A are called sites. The area of A, denoted by $|A|$, is the number of sites of A.

On Figure 1 are depicted directed animals on the three two-dimensional regular lattices: the square lattice, the triangular lattice, and the honeycomb lattice. The square lattice is the graph $\mathbb{N}^2$ with the natural edges $(i, j) \to (i+1, j)$ and $(i, j) \to (i, j+1)$ for all $i, j \geq 0$. The triangular lattice is simply obtained by adding the diagonal edges $(i, j) \to (i+1, j+1)$. Traditionally, the lattices are depicted so that every edge points upwards.

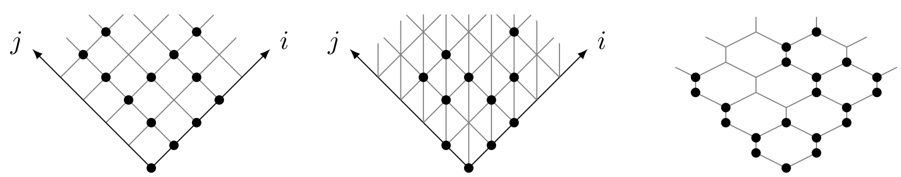


Figure 1: Single-source directed animals on the square, triangular and honeycomb lattices. All edges are directed upwards.

Single-source directed animals constitute a subclass of animals (an animal on a non-oriented graph Γ is simply a finite connected set of vertices of Γ). While the enumeration of animals on any lattice is an open problem despite extensive research for decades, directed animals are fairly easier to enumerate.

As we will not deal with general animals in this paper, we will abusively use the term animal instead of directed animal.

Single-source directed animals on the square and triangular lattices have been enumerated [6, 8, 2]. Specifically, let $a(n)$ and $\bar{a}(n)$ be the number of animals of source $\{(0,0)\}$ on the square and triangular lattice, respectively. The generating functions giving these numbers are:

$$\sum_{n\geq 1} a(n)t^n = \sum_A t^{|A|} = \frac{1}{2}\left(\sqrt{\frac{1+t}{1-3t}} - 1\right); \tag{1}$$

$$\sum_{n\geq 1} \bar{a}(n)t^n = \sum_A t^{|A|} = \frac{1}{2}\left(\frac{1}{\sqrt{1-4t}} - 1\right). \tag{2}$$

In each case, the second sum goes over all single-source animals on the appropriate lattice.

Even then, much remains unclear. The enumeration of directed animals on the honeycomb lattice is an open problem; on the square and triangular lattices, comparatively very little is known when one tries to take into account parameters other than area.

Today, two enumeration methods account for almost every known result on directed animals. One of them is the gas model technique, originally used by Dhar [6]. This technique was further developed and expanded by Bousquet-Mélou [3]; see also [11, 1] for more recent work.

The method used in this paper is the second one, based on a correspondence, due to Viennot [12], between animals and other objects called heaps of dominoes. The basic idea is to replace each site of an animal by a 2×1 domino, so that each domino either lies on the ground or sits on one or two other dominoes (Figure 2).

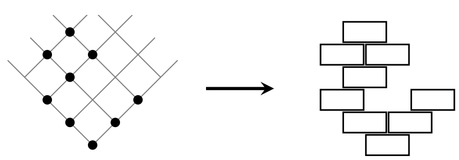


Figure 2: A directed animal on the square lattice can be turned into a heap by replacing each site by a 2×1 domino.

As we will see later, this method works for the triangular lattice as well. However, no simple model of heaps of dominoes has been found to correspond to animals on the honeycomb lattice. This may explain the lack of knowledge on the subject.

The purpose of this paper is to study three other parameters of directed animals, introduced below, and illustrated in Figure 3:

- two sites of an animal on the square or triangular lattice are adjacent if they are of the form $(i+1, j)$ and $(i, j+1)$. We denote by $j(A)$ the number of pairs of adjacent sites of A.

- a loop consists of two adjacent sites $(i+1,j)$ and $(i,j+1)$, along with a third site at $(i+1,j+1)$. We denote by $\ell(A)$ the number of loops of A.
- a neighbour of an animal A of source S is a vertex v not in A, such that $A \cup \{v\}$ is still a directed animal of source S. The number of neighbours of A is called the site perimeter of A and is denoted by $p(A)$.

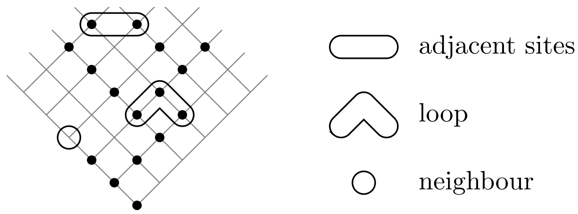


Figure 3: A directed animal on the square lattice with two adjacent sites, a loop, and a neighbour marked.

Taking, for instance, the site perimeter, we may consider the bivariate generating function counting single-source animals according to both area and perimeter on the square lattice:

$$A^p(t,u) = \sum_A t^{|A|} u^{p(A)}.$$

This generating function is not known, and is believed to be non-algebraic [9]. Instead, we will consider the generating function giving the total number of neighbours in the animals of a fixed area:

$$\sum_A p(A) t^{|A|} = \frac{\partial A^p}{\partial u}(t,1).$$

By dividing the total site perimeter of the animals of area n by the number of these animals, one gets the average site perimeter in animals of a fixed area. Alternatively, this generating function may be seen as counting single-source directed animals with a marked neighbour.

This function, and the ones that similarly give the average number of adjacent sites and loops, turned out to be easier to derive. Specifically, the value of the generating function counting the total number of loops of single-source animals on the square lattice was obtained by Bousquet-Mélou using gas model methods [3]:

$$\sum_A \ell(A) t^{|A|} = \frac{1}{2}\left(1 - \frac{1-4t+t^2+4t^3}{\sqrt{1+t}(1-3t)^{3/2}}\right). \tag{3}$$

As for the total site perimeter on the square lattice, it was the object of a conjecture by Conway in 1996 [5]:

$$\sum_A p(A) t^{|A|} = \frac{1}{2t(1+t)}\left(-1+t+t^2+\frac{1-3t+2t^2+t^3-3t^4}{\sqrt{1+t}(1-3t)^{3/2}}\right). \tag{4}$$

Le Borgne [10] also conjectured the value of similar generating functions counting the site perimeter of animals on bounded lattices.

Finally, we show in this paper that the total number of adjacent sites is given by:

$$\sum_A j(A)t^{|A|} = \frac{1}{2t(1+t)}\left(1 - \frac{1-4t+t^2+4t^3}{\sqrt{1+t}(1-3t)^{3/2}}\right). \tag{5}$$

In Section 4, we prove these three identities using combinatorial methods. Our formulæ also instantiate in the case of animals of any one-line source, animals on the half- and bounded square lattices, and to a certain degree on the triangular lattices. All these lattices are defined in Section 3.

Knowing, say, the total site perimeter of single-source animals of area n on the square lattice, we get their average perimeter by dividing by the number $a(n)$ of these animals:

$$p(n) = \frac{1}{a(n)} \sum_{|A|=n} p(A).$$

This quantity may thus be computed using (1) and (4).

The numbers of adjacent sites and loops are handled similarly. From these generating functions, singularity analysis [7] yields estimates on these quantities as n tends to infinity:

$$j(n) \sim \frac{n}{4}; \qquad \ell(n) \sim \frac{n}{9}; \qquad p(n) \sim \frac{3n}{4}.$$

The paper is organized as follows. In Section 2, we introduce in detail the notion of heaps of pieces and give several lemmas useful for animal enumeration. In Section 3, we enumerate directed animals of any source on several kinds of square and triangular lattices, according to area alone. In Section 4, we give a general method to derive the generating functions giving the average number of adjacent sites, number of loops, and site perimeter of directed animals on the square lattice; in Section 5, we give equivalents for most of these results in the triangular lattice. We illustrate our formulæ with a few examples in Section 6.

2 Heaps of pieces

2.1 Definitions

The notion of heaps of pieces, and its use to the enumeration of directed animals, is due to Viennot, and this topic is covered in much more detail in [12]. Intuitively, a heap is a finite set of pieces. It is built by dropping successively the pieces at certain positions, chosen from a given set. When the position of two pieces overlap, the second piece falls on the first, like in Figure 2. A more formal definition is given below.

Definition 2.1.1. Let Q be a set and $\mathcal{C}$ a reflexive symmetric relation on Q. A heap of the model $(Q, \mathcal{C})$ is a finite subset H of $Q \times \mathbb{N}^*$ verifying:

1. if (q, i) and (q', i) with $q \neq q'$ are in H, then (q, q') is not in $\mathcal{C}$;
2. if (q, i) is in H and $i > 1$, then there exists $(q', i-1)$ in H such that (q, q') is in $\mathcal{C}$.

The relation $\mathcal{C}$ is called the concurrency relation, and two positions q and q' are concurrent if (q, q') is in $\mathcal{C}$ (in the above intuitive definition, this means that they overlap).

The elements of a heap are called pieces. If (q, i) is a piece of a heap, q is called its position and i its height. The pieces of height 1 are called minimal; the set of the positions of the minimal pieces of a heap H is called the base of H, and denoted by $b(H)$. A pyramid is a heap with only one minimal piece; a heap is trivial if all its pieces are minimal. As the height of the pieces of a trivial heap is meaningless, we will often confuse a trivial heap with the set of the positions of its pieces. In this way, a trivial heap is also a finite set of pairwise nonconcurrent positions.

A piece (q, i) is said to sit on another piece of the form $(q', i-1)$ if q and q' are concurrent. Thus, in Definition 2.1.1, the second condition states that every non-minimal piece sits on at least one other piece. A finite set of pieces that verifies the first condition, but not necessarily the second is called a pre-heap.

The reason we study heaps of pieces is to provide an alternate vision of directed animals on the square and triangular lattices as heaps of certain models, called heaps of dominoes. In these models, the families of heaps and pre-heaps defined below will correspond to directed animals on the square and triangular lattice, respectively.

Definition 2.1.2. A heap (or pre-heap) H is strict if no piece sits on another piece with the same position.

Definition 2.1.3. An inflated heap is a strict pre-heap H, such that for every piece (q, i) in H such that $i > 1$, at least one of the following is in H:

- either a piece $(q', i-1)$ such that $q \neq q'$ and q and q' are concurrent;
- or the piece $(q, i-2)$.

With this definition, some non-minimal pieces of an inflated heap H may not sit on any other piece. We call these pieces unsupported; for reasons that will be clear when we link inflated heaps to animals on the triangular lattice, we denote by $c(H)$ the number of unsupported pieces of H.

Let H be a heap. A stack of pieces of H is a maximal set of pieces at the same position, sitting on each other (in other words, of the form (q, k) for $i \leq k \leq j$). By applying the operations depicted in Figure 4 to each stack of a heap, we may link both strict and inflated heaps to general heaps.

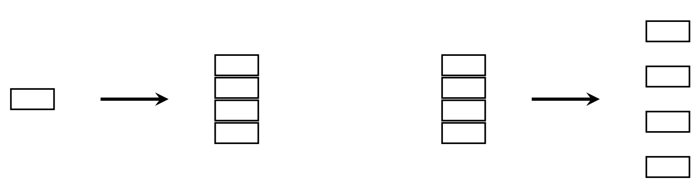


Figure 4: A general heap may be obtained by replacing each piece of a strict heap by a stack of pieces; it can be turned into an inflated heap by inflating each stack. In an inflated heap, only the lowest piece of each stack is not unsupported.

2.2 Generating functions

Now, we set some notations for the generating functions counting heaps of pieces. Let $(Q, \mathcal{C})$ be a model of heaps and S a set of positions.

Throughout this paper, we will use the same typographic pattern to denote generating functions counting heaps: the generating functions counting general heaps are denoted by calligraphic letters, while the ones counting strict heaps are denoted by standard capital letters. Moreover, the subscript S denotes heaps with base S, while the subscript $[S]$ denotes heaps with base included in S. In this way, define the four generating functions:

$$\mathcal{H}_S(t) = \sum_{b(H)=S} t^{|H|}; \qquad H_S(t) = \sum_{\substack{b(H)=S \\ H \text{ strict}}} t^{|H|};$$

$$\mathcal{H}_{[S]}(t) = \sum_{b(H)\subseteq S} t^{|H|}; \qquad H_{[S]}(t) = \sum_{\substack{b(H)\subseteq S \\ H \text{ strict}}} t^{|H|}.$$

The generating functions counting heaps of base S and included in S are obviously linked by

$$\mathcal{H}_{[S]}(t) = \sum_{\substack{T\subseteq S \\ T \text{ trivial}}} \mathcal{H}_T(t). \tag{6}$$

To get the value of $\mathcal{H}_S(t)$ as a function of the $\mathcal{H}_{[T]}(t)$'s, we may use the inclusion-exclusion principle:

$$\mathcal{H}_S(t) = \sum_{T\subseteq S} (-1)^{|S\smallsetminus T|} \mathcal{H}_{[T]}(t). \tag{7}$$

Alternatively, define the neighbourhood of S, denoted by $v(S)$ to be the set of positions concurrent to at least one position of S. Let H be a heap of base S. Then H consists of the pieces of S surmounted by a heap H', such that every minimal piece of H' sits on a piece of S. Thus, we have:

$$\mathcal{H}_S(t) = t^{|S|} \mathcal{H}_{[v(S)]}(t). \tag{8}$$

Identities similar to (6) and (7) also hold between generating functions counting strict heaps.

We now translate the correspondences depicted in Figure 4 in terms of generating functions. Replacing each piece by a nonempty stack is equivalent to the substitution $t \mapsto \frac{t}{1-t}$. We thus get the following link:

$$\mathcal{H}_S(t) = H_S\left(\frac{t}{1-t}\right). \tag{9}$$

Or, equivalently:

$$H_S(t) = \mathcal{H}_S\left(\frac{t}{1+t}\right). \tag{10}$$

Moreover, "inflating" a heap does not change its number of pieces, so that the generating functions counting general and inflated heaps are the same. Now,

let $\mathcal{H}_S^c(t,u)$ be the bivariate generating function counting inflated heaps, taking into account the number of unsupported pieces:

$$\mathcal{H}_S^c(t,u) = \sum_{b(H)=S} t^{|H|} u^{c(H)}.$$

As only the lowest piece of each stack of an inflated heap is not unsupported, this generating function is given by:

$$\mathcal{H}_S^c(t,u) = H_S\left(\frac{t}{1-tu}\right). \tag{11}$$

We may now compute the generating function counting the total number of unsupported pieces of the inflated heaps with a given number of pieces:

$$\sum_{b(H)=S} c(H) t^{|H|} = \frac{\partial \mathcal{H}_S^c}{\partial u}(t,1) = \frac{t^2}{(1-t)^2} H_S'\left(\frac{t}{1-t}\right).$$

Using (9), this is equivalent to:

$$\sum_{b(H)=S} c(H) t^{|H|} = t^2 \mathcal{H}_S'(t). \tag{12}$$

The case of heaps of base included in S is identical.

2.3 The Inversion Lemma

We say that a model $(Q,\mathcal{C})$ is finite if the set of positions Q is finite. In this case, the heaps of any given base may be enumerated using the result below, due to Viennot [12].

Let $(Q,\mathcal{C})$ be a finite model and S a subset of Q. Define the following generating function, counting trivial heaps with positions included in S, giving a weight $-t$ to each piece:

$$\mathcal{T}_{[S]}(t) = \sum_{\substack{T \subseteq S \\ T \text{ trivial}}} (-t)^{|T|}.$$

As Q is finite, there are only finitely many trivial heaps included in S, and thus, this generating function is a polynomial, which is usually relatively easy to compute.

Lemma 2.3.1 (Inversion Lemma). Let $(Q,\mathcal{C})$ be a finite model, and let S be a subset of Q. The generating function $\mathcal{H}_{[S]}(t)$ exists and is given by

$$\mathcal{H}_{[S]}(t) = \frac{\mathcal{T}_{[Q \setminus S]}(t)}{\mathcal{T}_{[Q]}(t)}.$$

In this way, we see that the generating function $\mathcal{H}_{[S]}(t)$ is a quotient of two polynomials, and is therefore rational.

In fact, the Inversion Lemma is valid as soon as the generating function $\mathcal{T}_{[Q]}(t)$ exists. It may exist even if Q is infinite, by giving the positions weights other than t, but we do not need it in this paper.

2.4 Monoid structure on heaps

The main advantage of regarding directed animals as heaps of dominoes is the extra structure heaps provide. This structure is much more difficult to see on directed animals. To begin with, let H be a heap of a model $(Q, \mathcal{C})$: then H is equipped with the following poset structure.

Definition 2.4.1. Let H be a heap. Define the relation $\prec$ on the pieces of H by $(q, i) \prec (q', i')$ if and only if q and q' are concurrent and $i < i'$. Define the partial order $\leq$ to be the reflexive transitive closure of $\prec$. A piece x is said to be below a piece y (or y is above x) if we have $x \leq y$.

This definition can be viewed more intuitively: let x be a piece of a heap H. Now, take the piece x and pull it upwards: the pieces that come up along are exactly the pieces above x.

The minimal pieces of a heap H, as defined above, are actually those that are minimal with respect to $\leq$; those that are maximal for $\leq$ are likewise called the maximal pieces of H. The set the maximal pieces of H is denoted by $m(H)$. We say that two pieces of a heap are independent if neither is above the other.

By turning a heap H upside down, letting all pieces fall in place, we obtain the dual heap $\tilde{H}$. This heap has the following property: a piece x of $\tilde{H}$ is below another piece y if and only if x is above y in the heap H. In particular, the minimal pieces of $\tilde{H}$ are the maximal pieces of H and vice versa.

Let now H be a heap and q a position. The heap denoted by $H \cdot q$ is built by dropping a piece at position q on top of H. Formally, $H \cdot q$ is equal to $H \cup \{(q, i)\}$ where i is the largest integer such that this is a heap. In this way, a heap may be built by dropping one piece at a time. This leads to the definition of the following operation on heaps.

Definition 2.4.2. Let H_1 and H_2 be two heaps. The product $H_1 \cdot H_2$ is built by dropping one by one all the pieces of H_2 in any order compatible with $\leq$.

The set of heaps of a model is a monoid under this law. Actually, this monoid is isomorphic to the partially commutative monoid on the alphabet Q with the concurrency relation $\mathcal{C}$. An illustration of this product is given in Figure 5.

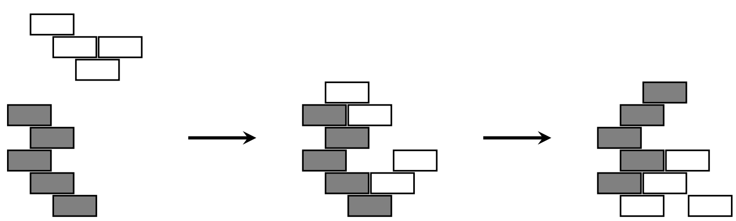


Figure 5: The product of two pyramids of dominoes, and the dual heap of the product.

This structure is compatible with duality: let H_1 and H_2 be two heaps. Then the dual heap of the product $H_1 \cdot H_2$ is $\tilde{H}_2 \cdot \tilde{H}_1$.

2.5 Factorized heaps

Definition 2.5.1. A factorized heap is a heap written as a product of two factors $(H_1 \cdot H_2)$. If $H_1 \cdot H_2 = H$, we say that $(H_1 \cdot H_2)$ is a factorization of H.

Our goal is to give some results on factorized heaps which will be useful in the enumeration of animals. Specifically, if H_1 is a fixed heap and S a trivial heap such that the base of H_1 is a subset of S, we want to study factorized heaps of base S of the form $(H_1 \cdot H_2)$. To do this, we need to figure out the base of a factorized heap $(H_1 \cdot H_2)$.

Let H be a heap. Define the neighbourhood of H, denoted by $v(H)$, to be the set of all positions concurrent to at least one piece of H (this definition extends the definition of Section 2.2 on trivial heaps).

Lemma 2.5.2. Let H_1 and H_2 be two heaps. The base of the product $H_1 \cdot H_2$ is

$$b(H_1 \cdot H_2) = b(H_1) \cup \big(b(H_2) \setminus v(H_1)\big).$$

Proof. Obviously, the minimal pieces of H_1 are still minimal in $H_1 \cdot H_2$. As for the minimal pieces of H_2, those that are minimal in $H_1 \cdot H_2$ are those not above a piece of H_1, thus the ones at positions not in the neighbourhood of H_1. $\square$

Let now H_1 be a heap, and let S be a trivial heap verifying $b(H_1) \subseteq S$. Let $\mathcal{F}_{H_1|S}(t)$ be the generating function counting factorized heaps of the form $(H_1 \cdot H_2)$, such that $b(H_1 \cdot H_2) = S$.

Lemma 2.5.3. Let H_1 and H_1' be two heaps sharing the same base and neighbourhood, and let S be a trivial heap such that $b(H_1) \subseteq S$. The following identity holds:

$$\mathcal{F}_{H_1'|S}(t) = t^{|H_1'|-|H_1|}\mathcal{F}_{H_1|S}(t).$$

Proof. Consider the mapping $(H_1 \cdot H_2) \mapsto (H_1' \cdot H_2)$. As H_1 and H_1' share the same base and neighbourhood, Lemma 2.5.2 implies that it maps heaps of base S to heaps of base S. Moreover, we clearly have $|H_1' \cdot H_2| = |H_1 \cdot H_2| + |H_1'| - |H_1|$. The lemma follows. $\square$

We now consider the case of strict heaps. A direct analogue of Lemma 2.5.3 does not hold, since the product of two strict heaps might not be strict. Instead, we use the following intermediate heaps.

Definition 2.5.4. A factorized heap $(H_1 \cdot H_2)$ is strict if $H_1 \cdot H_2$ is strict; it is almost-strict if H_1 and H_2 are both strict.

Let H_1 be a strict heap, and let S be a trivial heap containing $b(H_1)$. We denote by $F_{H_1|S}(t)$ the generating function counting strict factorized heaps of the form $(H_1 \cdot H_2)$, with base included in S. We also denote by $F^*_{H_1|S}(t)$ its analogue counting almost-strict factorized heaps.

Lemma 2.5.5. Let H_1 be a strict heap, and S a trivial heap containing $b(H_1)$. The generating functions counting strict and almost-strict factorized heaps of base S of the form $(H_1 \cdot H_2)$ are linked as follows:

$$F^*_{H_1|S}(t) = (1+t)^{|m(H_1)|}F_{H_1|S}(t).$$

Proof. Let $(H_1 \cdot H_2)$ be an almost-strict factorized heap of base S. Then, in the underlying heap $H_1 \cdot H_2$, the only pieces that might sit on another one at the same position are the minimal pieces of H_2 sitting on maximal pieces of H_1.

Let H_2' be the heap formed by removing from H_2 all minimal pieces at the same position as a maximal piece of H_1. Then $(H_1 \cdot H_2')$ is a strict factorized heap, and its base is still S.

In this way, an almost-strict factorized heap is built from a strict factorized heap $(H_1 \cdot H_2')$ by possibly adding a piece under H_2' at the same position as each maximal piece of H_1. Each such piece "possibly added" accounts for a $(1+t)$ factor in the generating function $F^*_{H_1|S}$, hence the formula. □

Lemma 2.5.6. Let H_1 and H_1' be two strict heaps sharing the same base and neighbourhood, and let S be a trivial heap such that $b(H_1) \subseteq S$. The following identities hold:

$$F^*_{H_1'|S}(t) = t^{|H_1'|-|H_1|} F^*_{H_1|S}(t);$$
$$F_{H_1'|S}(t) = \frac{t^{|H_1'|-|H_1|}}{(1+t)^{|m(H_1')|-|m(H_1)|}} F_{H_1|S}(t).$$

Proof. The proof of the first identity is similar to that of Lemma 2.5.3: as H_1 and H_1' have the same base and neighbourhood, Lemma 2.5.2 entails that the mapping $(H_1 \cdot H_2) \mapsto (H_1' \cdot H_2)$ is a bijection between the desired sets of almost-strict factorized heaps. The second identity is then a consequence of Lemma 2.5.5. □

2.6 Heaps marked with a set of pieces

We are now concerned with heaps with a set of pieces marked. These objects intervene naturally in the enumeration of animals (for example, computing the average number of adjacent sites of animals of a given area is equivalent to enumerate animals with two adjacent sites marked). We define two mappings that link these heaps with factorized heaps.

Definition 2.6.1. Let H be a heap marked with a set of pieces X. Define $\Theta(H, X)$ to be the factorization $(H_1 \cdot H_2)$ of H such that H_1 is the heap formed of all pieces below at least one piece of X (including the pieces of X). Symmetrically, define $\Omega(H, X)$ to be the factorization $(H_1 \cdot H_2)$ of H such that H_2 is the heap formed of all pieces above at least one piece of X.

More intuitively, the factorization $\Theta(H, X)$ is formed by taking the pieces of X and pulling them downwards, splitting the heap H into the desired two parts, while $\Omega(H, X)$ is formed by pulling them upwards. Equivalently, $\Omega(H, X)$ may be defined as the dual factorized heap of $\Theta(\tilde{H}, X)$.

We now consider the special case where the pieces of X are pairwise independent.

Lemma 2.6.2. Both mappings Θ and Ω are bijections from the set of all heaps marked with a set of pairwise independent pieces to the set of all factorized heaps. Furthermore, let $(H_1 \cdot H_2)$ be a factorized heap. Its inverse image by Θ is obtained by marking the set X of the maximal pieces of H_1; its inverse image by Ω is obtained by marking the set of minimal pieces of H_2.

Proof. Let (H, X) be a heap marked with a set of pairwise independent pieces, and let $(H_1 \cdot H_2)$ be a factorization of H. We will prove that $\Theta(H, X) = (H_1 \cdot H_2)$ is equivalent to $m(H_1) = X$.

Assume that $\Theta(H, X) = (H_1 \cdot H_2)$. Let x be a maximal piece of H_1. As x is in H_1, it is below one piece of X; as it is maximal, it is below no other piece of H_1 than itself. Therefore, x must be in X. Let now y be a non-maximal piece of H_1: then it is below a maximal piece. As the elements of X are pairwise independent, y cannot be in X, so we have $m(H_1) = X$.

Conversely, assume that $m(H_1) = X$. Every piece of H_1 is below a maximal piece, and no piece of H_2 can be below a piece of H_1 in $H_1 \cdot H_2$. Therefore, we have $\Theta(H, X) = (H_1 \cdot H_2)$.

The case of Ω is easily obtained by duality. $\square$

In an enumerative context, these bijections are particularly useful, because the image of a marked heap H is a factorization of H. In particular, any heap has the same number of pieces as its image by Θ or Ω (in fact, for every position p, the number of pieces at position p is also preserved). Thus, these bijections easily translate into identities between generating functions.

Finally, we give a first practical application of the above results. Let S be a set of positions and q a position. Define the following two generating functions:

- let $\mathcal{H}_{[S]}^{(q)}(t)$ be the generating function counting heaps of base included in S, marked with a piece at position q;
- let $\mathcal{V}_{[S]}^{q}(t)$ be the generating function counting heaps of base included in S avoiding q, that is, such that no piece is concurrent to q (in other words, they are the heaps of base included in S of the model $(Q \smallsetminus v(q), \mathcal{C})$).

As usual, we use the same notations with standard capital letters to denote generating functions counting strict heaps.

Proposition 2.6.3. The generating function counting heaps of base included in S marked with a piece at position q is given by:

$$\mathcal{H}_{[S]}^{(q)}(t) = \begin{cases} \mathcal{H}_{[S]}(t)\mathcal{H}_{\{q\}}(t) & \text{if } q \in S, \\ \Big(\mathcal{H}_{[S]}(t) - \mathcal{V}_{[S]}^{q}(t)\Big)\mathcal{H}_{\{q\}}(t) & \text{otherwise;} \end{cases}$$

$$H_{[S]}^{(q)}(t) = \frac{1}{1+t}\begin{cases} H_{[S]}(t)H_{\{q\}}(t) & \text{if } q \in S, \\ \Big(H_{[S]}(t) - V_{[S]}^{q}(t)\Big)H_{\{q\}}(t) & \text{otherwise,} \end{cases}$$

where $\mathcal{H}_{\{p\}}(t)$ counts all pyramids of base $\{p\}$.

Proof. Let H be a heap marked with a piece x at position q. We factorize the heap H by using the bijection Ω, pulling the piece x upwards: we thus have $H = H_1 \cdot H_2$, where H_2 is the pyramid composed of all pieces of H above x.

According to Lemma 2.5.2, the base of H is $b(H_1)$ if q is concurrent to a piece of H_1, and $b(H_1) \cup \{q\}$ otherwise. Thus, if q is in S then the base of H_1 must simply be included in S; if q is not in S, then the base of H_1 must be included in S and a piece of H_1 must be concurrent to q. We therefore get the value of $\mathcal{H}_{[S]}^{(q)}(t)$.

Let now $H_{[S]}^{(q)*}(t)$ be the generating function counting strict heaps of base included in S, with a piece at position q marked, such that the marked piece alone may sit on a piece at the same position. By factorizing such a heap in the

same way, one gets an almost-strict factorized heap to which we may apply the same reasoning. Moreover, we have

$$H_{[S]}^{(q)*}(t) = (1+t)H_{[S]}^{(q)}(t),$$

as just one piece may be duplicated. This completes the proof. $\square$

3 Directed animals and heaps of dominoes

In this section, we will use the results on heaps of pieces to compute the generating functions counting directed animals on various square and triangular lattices, according to area alone. To do this, we will define some models of heaps, called heaps of dominoes, and link them to directed animals.

In Section 1, we presented the full square and triangular lattices, which have the set of vertices $\mathbb{N}^2$ (examples of animals on these lattices are shown in Figure 1). We now define variants of these lattices, delimited by vertical lines.

The half-lattice is delimited by one vertical line, while the rectangular lattices are constrained between two vertical lines. The cylindrical lattices are obtained by taking two vertical lines and gluing them together.

The rectangular and cylindrical lattices are called bounded, as there is a bounded number of vertices on each horizontal line. The distance between the contraining vertical lines, which is always even in the cylindrical case, is called the width of the lattice (Figure 6).

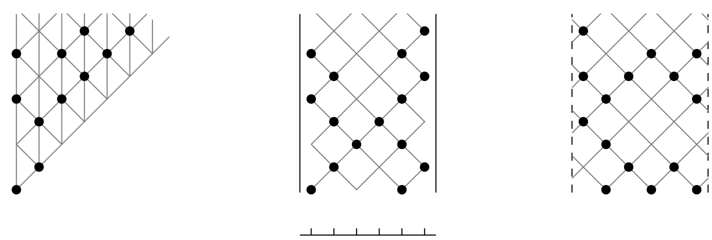


Figure 6: Animals on the triangular half-lattice, and the square rectangular and cylindrical lattices of width 6.

In addition, we restrain ourselves to animals with a one-line source, that is, a source which is a finite subset of a horizontal line.

We denote by $A(t)$ the generating function counting single-source animals on the full square lattice, and by $D(t)$ the one counting single-source half-animals on the square lattice, that is, animals on the half-lattice with the source at the corner of the lattice. Moreover, let $A_{2m}(t)$ be the generating function counting single-source animals on the cylindrical square lattice of width $2m$, and $D_m(t)$ the one counting single-source half-animals on the rectangular square lattice of width m.

In a way similar to heaps, we also use the notations $\mathcal{A}(t)$, $\mathcal{D}(t)$, $\mathcal{A}_{2m}(t)$ and $\mathcal{D}_m(t)$ to denote the generating functions counting the same animals on the various triangular lattices.

3.1 Heaps of dominoes

As stated before, we will compute these generating functions by regarding directed animals as heaps of dominoes. The model we consider depends on the kind of lattice:

- the model $Q = \mathbb{Z}$ corresponds to the full lattice;
- the model $Q = \mathbb{N}$ corresponds to the half-lattice;
- the model $Q = [0, m-1]$ corresponds to the rectangular lattice of width m;
- the model $Q = \mathbb{Z}/2m\mathbb{Z}$ corresponds to the cylindrical lattice of width $2m$.

In each case, two positions are concurrent if they are at distance at most 1:

$$\mathcal{C} = \big\{(q, q')\big||q - q'| \leq 1\big\}.$$

The strict heaps of these models are naturally in bijection with animals on the square lattices, while the inflated heaps are in bijection with animals on the triangular lattices. These bijections are illustrated in Figure 7.

Let Γ be a square lattice, and Δ the associated triangular lattice. Let S be a one-line source: in terms of heaps of dominoes, S becomes a trivial heap, which we abusively also denote by S. Animals of source S on the lattices Γ and Δ become strict and inflated heaps of base S, respectively. Using the notations of Section 2.2, they are therefore counted respectively by $H_S(t)$ and $\mathcal{H}_S(t)$.

Definition 3.1.1. Let A be an animal of source S on the lattice Δ, and let (i, j) be a site of A outside the source. We say that (i, j) is only supported at the center if there are no sites of A at $(i - 1, j)$ or $(i, j - 1)$; we denote by $c(A)$ the number of sites of A only supported at the center.

Let H be the heap of dominoes corresponding to A; a site v of A is only supported at the center if and only if the corresponding domino of H is unsupported. This enables us to use the results of Section 2.2.

The heaps of dominoes obtained from animals also have an interesting property which we will exploit later.

Definition 3.1.2. A (pre-)heap of dominoes H is aligned if the quantities $q + i$, for all pieces (q, i) of H, have the same parity.

Let S be a one-line source. By construction, the trivial heap corresponding to S is always aligned. Moreover, the following result is easy to check by induction on the height of H:

Lemma 3.1.3. Let H be a strict heap or inflated heap of dominoes such that its base $b(H)$ is aligned. Then H is aligned.

Thus, we conclude that the heaps of dominoes corresponding to directed animals on the square and triangular lattices are always aligned.

In the next subsections, we give ways to compute the generating functions $H_S(t)$ and $\mathcal{H}_S(t)$ counting animals of source S on the lattices Γ and Δ. We denote by $a(n)$ and $\bar{a}(n)$ the number of directed animals of source S on these two lattices, so that:

$$H_S(t) = \sum_{n \geq 0} a(n)t^n; \qquad \mathcal{H}_S(t) = \sum_{n \geq 0} \bar{a}(n)t^n.$$

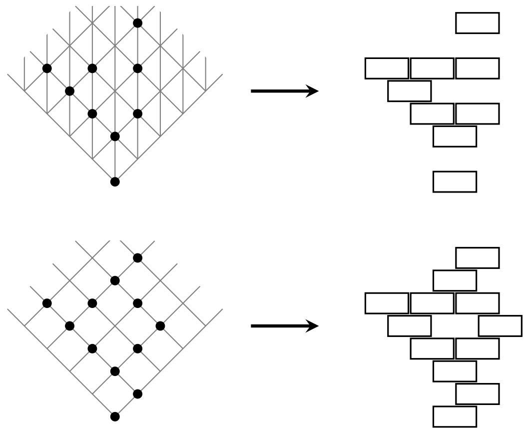


Figure 7: By replacing each site by a domino, animals on the triangular lattice become inflated heaps while those on the square lattice become strict heaps. In the animal at the top, two sites are only supported at the center.

3.2 Bounded lattices

Now, assume that Γ is a bounded lattice, either rectangular or cylindrical. Then as the corresponding model $(Q, \mathcal{C})$ of heaps of dominoes is finite, the Inversion Lemma (Lemma 2.3.1) enables us to compute the generating functions counting the heaps of that model.

Let $T_m(t)$ be the polynomial counting trivial heaps of the model $[0, m-1]$, each piece having weight $-t$. Likewise, let $\widehat{T}_{2m}(t)$ be the polynomial counting trivial heaps of the model $\mathbb{Z}/2m\mathbb{Z}$. The following proposition gives an inductive formula to compute these polynomials.

Proposition 3.2.1. The polynomials counting trivial heaps of dominoes on the finite models with each piece having weight $-t$ are given by:

$$\begin{aligned} T_m(t) &= 1 && \text{if } m \leq 0; \\ T_m(t) &= T_{m-1}(t) - tT_{m-2}(t) && \text{otherwise;} \\ \widehat{T}_{2m}(t) &= T_{2m-1}(t) - tT_{2m-3}(t). \end{aligned}$$

Actually, these polynomials are well-known: they are closely related to the Fibonacci (or Chebyshev) polynomials.

Sketch of the proof. Let T be a trivial heap of the model $[0, m-1]$: then T is counted by $T_{m-1}(t)$ if $m-1$ is not in T, and by $-tT_{m-2}(t)$ otherwise. The cylindrical case is handled similarly. □

Using this result and the Inversion Lemma, we are able to derive the values of the generating functions counting general (or equivalently, inflated) heaps of dominoes of any base, hence animals on the triangular bounded lattices. For example, we give below the generating functions counting single-source half-animals and animals.

Proposition 3.2.2. The generating functions for single-source animals on the cylindrical triangular lattice and half-animals on the rectangular triangular lattice are:

$$\mathcal{A}_{2m}(t) = \frac{T_{2m-1}(t)}{\widehat{T}_{2m}(t)} - 1; \tag{13}$$

$$\mathcal{D}_m(t) = \frac{T_{m-1}(t)}{T_m(t)} - 1. \tag{14}$$

The generating functions counting the corresponding animals on the square lattices are found by replacing t by $\frac{t}{1+t}$. Using this result, we are able to derive the asymptotic behaviour of the number of animals of area n as n tends to infinity.

Lemma 3.2.3. For all $m \geq 0$, The polynomials $T_m(t)$ and $\widehat{T}_{2m}(t)$ have only real, simple roots. Let ρ_m and σ_{2m} be their smallest respective roots. The sequences (ρ_m) and (σ_{2m}) are decreasing and tend to $1/4$ as m tends to infinity; moreover, we have $\sigma_{2m} \leq \rho_{2m}$ for all m.

Proof. First, we check by induction using the definitions of $T_m(t)$ and $\widehat{T}_{2m}(t)$ the following identities:

$$T_m\left(\frac{1}{4\cos^2\theta}\right) = \frac{\sin[(m+2)\theta]}{(2\cos\theta)^{m+1}\sin\theta};$$

$$\widehat{T}_{2m}\left(\frac{1}{4\cos^2\theta}\right) = \frac{2\cos(2m\theta)}{(2\cos\theta)^{2m}}.$$

We also check that the polynomials $T_m(t)$ and $\widehat{T}_{2m}(t)$ have degree $\lfloor \frac{m}{2} \rfloor$ and m, respectively. This proves that all roots of both polynomials are real and simple; moreover, we have:

$$\rho_m = \frac{1}{4\cos^2\frac{\pi}{m+2}};$$

$$\sigma_{2m} = \frac{1}{4\cos^2\frac{\pi}{4m}}.$$

The lemma follows from these values. □

Proposition 3.2.4. Assume that Γ is a bounded square lattice and let Δ be the associated triangular lattice. Let S be a one-line source. As n tends to infinity, the number of animals of source S and area n on the lattices Γ and Δ are asymptotically of the form:

$$a(n) \sim \lambda\mu^n; \qquad \bar{a}(n) \sim \bar{\lambda}\bar{\mu}^n.$$

Moreover, while the constants λ and $\bar{\lambda}$ depend on the source, μ depends only on the lattice, and $\bar{\mu}$ is equal to $\mu + 1$.

In particular, if S and S' are two one-line sources, the number of animals of area n with source S and S' on any bounded lattice differ only by a multiplicative constant.

Proof. Assume that Γ is the rectangular lattice of width m. We derive the generating function $\mathcal{H}_{[S]}(t)$ using the Inversion Lemma: its numerator is a product of factors $T_{k_i}(t)$, for some $k_i < m$, and its denominator is $T_m(t)$. We then use singularity analysis: Lemma 3.2.3 entails that the smallest pole of $\mathcal{H}_{[S]}(t)$ is ρ_m, and that this pole is simple. We thus get the estimate for $\bar{a}_n$, with $\bar{\mu} = 1/\rho_m$.

The case of the cylindrical model of width $2m$ is identical, with the denominator being $\widehat{T}_{2m}(t)$. As all ρ_k for $k < 2m$ are greater than σ_{2m} according to Lemma 3.2.3, we find $\bar{\mu} = 1/\sigma_{2m}$.

The generating function $H_{[S]}(t)$ is obtained by substituting t by $\frac{t}{1+t}$. Thus, the smallest pole $1/\mu$ of $H_{[S]}(t)$ verifies

$$\frac{1/\mu}{1+1/\mu} = \frac{1}{\bar{\mu}}$$

which yields $\bar{\mu} = \mu + 1$. As $1/\bar{\mu}$ is a simple pole of $\mathcal{H}_{[S]}(t)$, this pole is also simple.

The case of animals with source S is handled with the inclusion-exclusion principle. $\square$

3.3 Unbounded lattices

Now, we deal with the case where Γ is an unbounded lattice; as usual, we denote by Δ the associated triangular lattice. The generating functions counting animals and half-animals on the square and triangular unbounded lattices are given below. Various ways to derive them can be found in [6, 8, 2].

Proposition 3.3.1. The generating functions counting half-animals and animals on the unbounded square and triangular lattices are given by:

$$D(t) = \frac{1 - t - \sqrt{(1+t)(1-3t)}}{2t}; \tag{15}$$

$$\mathcal{D}(t) = \frac{1 - \sqrt{1-4t}}{2t} - 1; \tag{16}$$

$$A(t) = \frac{1}{2}\left(\sqrt{\frac{1+t}{1-3t}} - 1\right); \tag{17}$$

$$\mathcal{A}(t) = \frac{1}{2}\left(\frac{1}{\sqrt{1-4t}} - 1\right). \tag{18}$$

Of course, as usual, the generating function counting animals on the square and triangular lattices are linked by:

$$\mathcal{D}(t) = D\left(\frac{t}{1-t}\right);$$
$$\mathcal{A}(t) = A\left(\frac{t}{1-t}\right).$$

Moreover, we have, as m tends to infinity, the following limits in the space of formal power series:

$$\mathcal{D}_m(t) \to \mathcal{D}(t);$$
$$\mathcal{A}_{2m}(t) \to \mathcal{A}(t).$$

If S is a one-line source, we will now compute the generating functions counting animals of source S on the lattices Γ and Δ. To do so, we see these animals as strict and inflated heaps of base S on the corresponding model Q of heaps of dominoes.

In the next propositions, we give only the generating function counting general, or inflated, heaps of base S. To get the one counting strict heaps, it suffices to replace t by $\frac{t}{1+t}$. Thus, the generating functions $\mathcal{D}(t)$ and $\mathcal{A}(t)$ are replaced by their non-calligraphic counterparts.

In the model $Q = \mathbb{Z}$, for any position q, the pyramids of base $\{q\}$ are counted by the generating function $\mathcal{A}(t)$. In the model $Q = \mathbb{N}$, this is not the case, as shown by the following result.

Lemma 3.3.2. Let $Q = \mathbb{N}$ be the model corresponding to the half-lattice, and let q be a position. The generating function counting pyramids of base q is given by

$$\mathcal{H}_{\{q\}}(t) = \frac{1 - \mathcal{D}(t)^{q+1}}{1 - \mathcal{D}(t)} \mathcal{D}(t).$$

Regarded as directed animals, these pyramids are more commonly called "directed animals with left width at most q". A proof of the above formula is found in [2].

We call a trivial heap compact if it is of the form $\{q, q+2, \ldots, q+2k-2\}$, with $k \geq 1$. The heaps with a compact base are enumerated by the following proposition.

Lemma 3.3.3. Let $Q = \mathbb{Z}$ and let C_k be a compact trivial heap with k pieces. The generating function counting heaps with base C_k is given by:

$$\mathcal{H}_{C_k}(t) = \mathcal{A}(t)\mathcal{D}(t)^{k-1}.$$

Let now $Q = \mathbb{N}$ and let $C_{q,k}$ be the compact trivial heap $\{q, q+2, \ldots, q+2k-2\}$. The generating function counting heaps with base $C_{q,k}$ is given by:

$$\mathcal{H}_{C_{q,k}}(t) = \frac{1 - \mathcal{D}(t)^{q+1}}{1 - \mathcal{D}(t)} \mathcal{D}(t)^k.$$

By summing the first identity over all $k \geq 1$, we get this surprisingly simple result, also found in [8].

Proposition 3.3.4. The number of directed animals with any compact source on the square lattice with area n is 3^{n-1}. On the triangular lattice, this number is 4^{n-1}.

Proof of Lemma 3.3.3. To prove this result, we use the construction depicted in Figure 8, due to Bétréma and Penaud [2].

In this way, a heap with base $C_{q,k}$ can be seen as the product of $k-1$ half-pyramids and one pyramid with base $\{q\}$. This yields the announced generating functions. □

Finally, we are able to compute the generating function counting heaps of base S, for any aligned trivial heap S.

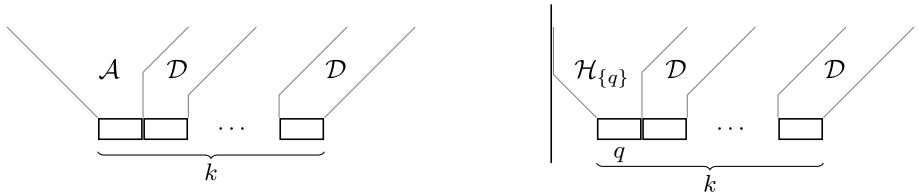


Figure 8: A heap with a compact base with k dominoes breaks into $k-1$ half-pyramids and one pyramid.

Proposition 3.3.5. Let Q be either infinite model of heaps of dominoes ($Q = \mathbb{Z}$ or $Q = \mathbb{N}$), and let S be an aligned trivial heap. Let C be the smallest compact trivial heap such that $S \subseteq C$. The generating function counting heaps with base S is given by:

$$\mathcal{H}_S(t) = \frac{\mathcal{T}_{[v(C)\smallsetminus v(S)]}(t)}{t^{|C|-|S|}}\mathcal{H}_C(t)$$

where $\mathcal{T}_{[v(C)\smallsetminus v(S)]}(t)$ counts all trivial heaps included in $v(C) \smallsetminus v(S)$, each piece having weight $-t$.

In the case of the full lattice, this result is found in an equivalent form in [1].

Proof. First, define the finite model Q_m in the following manner: if $Q = \mathbb{Z}$, let Q_m be the model $\mathbb{Z}/2m\mathbb{Z}$; if $Q = \mathbb{N}$, let $Q_m = [0, m-1]$. In either case, m is assumed to be large enough so that S and C are still trivial heaps of the model $(Q_m, \mathcal{C})$.

Using identity (8) and the Inversion Lemma, we compute the generating functions counting heaps of this model with base S and C:

$$\mathcal{H}_{S,m}(t) = t^{|S|}\mathcal{H}_{[v(S)],m}(t) = t^{|S|}\frac{\mathcal{T}_{[Q_m\smallsetminus v(S)]}(t)}{\mathcal{T}_{[Q_m]}(t)};$$
$$\mathcal{H}_{C,m}(t) = t^{|C|}\mathcal{H}_{[v(C)],m}(t) = t^{|S|}\frac{\mathcal{T}_{[Q_m\smallsetminus v(C)]}(t)}{\mathcal{T}_{[Q_m]}(t)}.$$

As S is included in C, $v(S)$ is also included in $v(C)$, and therefore, we can write:

$$Q_m \smallsetminus v(S) = \big(Q_m \smallsetminus v(C)\big) \cup \big(v(C) \smallsetminus v(S)\big).$$

As C is a compact source, $v(C)$ is a segment of integers, so that the only positions of $v(C)$ concurrent to a positions of $Q_m \smallsetminus v(C)$ are its endpoints. As C is the smallest compact source containing S, both endpoints of $v(C)$ are also in $v(S)$. Therefore, no position of $Q_m \smallsetminus v(C)$ is concurrent to a position of $v(C) \smallsetminus v(S)$.

Thus, we see that choosing a trivial heap outside $v(S)$ is equivalent to choosing independently two trivial heaps, one outside $v(C)$ and the other in $v(C) \smallsetminus v(S)$. In terms of generating functions, we have:

$$\mathcal{T}_{[Q_m\smallsetminus v(S)]}(t) = \mathcal{T}_{[Q_m\smallsetminus v(C)]}(t)\mathcal{T}_{[v(C)\smallsetminus v(S)]}(t).$$

Therefore, we get:

$$\frac{\mathcal{H}_{S,m}(t)}{\mathcal{H}_{C,m}(t)} = \frac{t^{|S|}}{t^{|C|}}\mathcal{T}_{[v(C)\smallsetminus v(S)]}(t).$$

We conclude by letting m tend to infinity. $\square$

Similarly to the case of bounded lattices, we get the asymptotic behaviour of the numbers $a(n)$ and $\bar{a}(n)$ of directed animals of source S and area n on the lattices Γ and Δ.

Proposition 3.3.6. Assume that Γ is the full lattice, and let S be a one-line source. As n tends to infinity, we have:

$$a(n) \sim \lambda \frac{3^n}{\sqrt{n}}; \qquad \bar{a}(n) \sim \bar{\lambda} \frac{4^n}{\sqrt{n}}$$

where λ and $\bar{\lambda}$ are constants depending on the source S.

Assume now that Γ is the half-lattice. The number of animals of area n is equivalent to:

$$a(n) \sim \lambda \frac{3^n}{n^{3/2}}; \qquad \bar{a}(n) \sim \bar{\lambda} \frac{4^n}{n^{3/2}}$$

with λ and $\bar{\lambda}$ again depending on the source S.

Again, if S and S' are two one-line sources of a lattice, the number of animals of source S and S' of area n asymptotically differ only by a multiplicative constant.

Proof. Once again, we use singularity analysis. The smallest pole of the generating functions $\mathcal{A}(t)$ and $\mathcal{D}(t)$ is $t = 1/4$, and we have as t tends to $1/4$:

$$\mathcal{A}(t) = (1-4t)^{-1/2} + \mathrm{o}\big((1-4t)^{-1/2}\big);$$
$$\mathcal{D}(t) = 1 - 2\sqrt{1-4t} + \mathrm{o}\big(\sqrt{1-4t}\big).$$

Now, assume that $Q = \mathbb{Z}$ and let C_k be a compact source. According to Lemma 3.3.3, we have:

$$\mathcal{H}_{C_k}(t) = (1-4t)^{-1/2} + \mathrm{o}\big((1-4t)^{-1/2}\big).$$

If $Q = \mathbb{N}$, then let $C_{q,k}$ be a compact source. We have:

$$\mathcal{H}_{C_{q,k}}(t) = q + 1 - (q+1)(2k+q)\sqrt{1-4t} + \mathrm{o}\big(\sqrt{1-4t}\big).$$

Hence, we get the announced estimates if S is a compact source.

Let S be any one-line source, and let C be the smallest compact source containing S. We use Proposition 3.3.5 to link the generating functions $\mathcal{H}_S(t)$ and $\mathcal{H}_C(t)$. Thus, $t^{|C|-|S|}\mathcal{H}_S(t)/\mathcal{H}_C(t)$ is a polynomial counting trivial heaps of a finite model, which according to Lemma 3.2.3 has all roots larger than $1/4$. Hence, this polynomial accounts only for a multiplicative constant in the behaviour of the coefficients of $\mathcal{H}_S(t)$.

As in the proof of Proposition 3.2.4, the case of the square lattice is handled by performing the substitution $t \mapsto \frac{t}{1+t}$. □

4 Average number of adjacent pieces, number of loops, and site perimeter on the square lattice

4.1 Notations and results

In this section, Γ is a square lattice (full, half, cylindrical or rectangular), and S is a one-line source of Γ. In Section 3, we have computed the generating

functions $H_S(t)$ and $H_{[S]}(t)$ counting directed animals on Γ of source S and with a source included in S.

In Section 1, we have defined the number $j(A)$ of pairs of adjacent sites and $\ell(A)$ of loops of an animal A. Now, define the generating function, counting the total number of adjacent sites of animals of a given area:

$$J_S(t) = \sum_A j(A) t^{|A|}$$

where the sum goes over all animals of source S on Γ. In the same way, define the generating function $L_S(t)$ counting the total number of loops; also define the analogous generating functions $J_{[S]}(t)$ and $L_{[S]}(t)$ dealing with animals of source included in S.

Definition 4.1.1. Let A be an animal on Γ with a source included in S. A neighbour of A is a vertex v of Γ not in A such that $A \cup \{v\}$ is still a directed animal with a source included in S.

Moreover, assume that the graph Γ is embedded in a larger graph Γ'. An internal neighbour of A is a neighbour of A seen as an animal on Γ. An external neighbour of A is a neighbour of A seen as an animal on Γ'.

For the purpose of this definition, we regard the half-lattice and the rectangular lattices as embedded in the full lattice. The full and cylindrical lattices are not naturally embedded in any other graph than themselves, so in these lattices internal and external neighbours are identical.

We denote by $p_i(A)$ the number of internal neighbours of A (or internal site perimeter) and by $p_e(A)$ its number of external neighbours (or external site perimeter). Thus, we define the generating functions $P^i_S(t)$, $P^e_S(t)$, $P^i_{[S]}(t)$ and $P^e_{[S]}(t)$ in the same way as above.

To compute these generating functions, we again use the correspondence between animals and heaps of dominoes. We denote by $(Q, \mathcal{C})$ the model of heaps of dominoes associated to the lattice Γ; we also denote by S the aligned trivial heap associated to the source S.

In this section, we express all the above defined generating functions in terms of other generating functions that can all be computed using the results of Section 3. Namely:

- let $E_S(t)$ be the generating function counting strict heaps of base S marked with a piece at the edge of the model, i.e. at a position q such that either $q-1$ or $q+1$ is not in Q;
- let $U_S(t)$ be the generating function counting strict heaps of base S marked with a piece at a position q such that $q+2$ is not in Q;
- if $q-2$ is in S but q is not, let $W^q_S(t)$ be the generating function counting strict heaps of base $S \cup \{q\}$;
- let $W_S(t)$ be the sum of $W^q_S(t)$ for all q in $(S+2) \setminus S$.

Remark. As the full and cylindrical models have no edge, the generating function $E_S(t)$ is always zero in these models. Moreover, in all models but the rectangular, the position $q+2$ is always in Q if q is, so the generating function $U_S(t)$ is also zero.

In the same way, define the generating functions $E_{[S]}(t)$ and $U_{[S]}(t)$. Finally, define:

- if $q-2$ is in S but q is not, $W^q_{[S]}(t)$ is the generating function counting strict heaps of base included in $S \cup \{q\}$ containing $q-2$ and q;
- $W_{[S]}(t)$ is the sum of $W^q_{[S]}(t)$ for all q in $(S+2) \setminus S$.

The results of Section 3 enable us to compute the generating functions $W_S(t)$ and $W_{[S]}(t)$. Moreover, as there are at most three positions q such that $q-1$, $q+1$ or $q+2$ is not in Q, we can compute the generating functions $E_S(t)$, $U_S(t)$ and their counterparts using Proposition 2.6.3.

The following theorem, proved in Sections 4.2 and 4.3, thus gives the value of the generating functions counting the total number of adjacent sites, loops and site perimeter of directed animals on Γ.

Theorem 4.1.2. The generating functions counting the total number of adjacent pieces, loops and site perimeters of the animals with source included in S are given by:

$$J_{[S]}(t) = \frac{t^2 H'_{[S]}(t) - tU_{[S]}(t) - W_{[S]}(t)}{1+t}; \tag{19}$$

$$L_{[S]}(t) = t(1+t)J_{[S]}(t); \tag{20}$$

$$P^e_{[S]}(t) = |S|H_{[S]}(t) + tH'_{[S]}(t) - J_{[S]}(t); \tag{21}$$

$$P^i_{[S]}(t) = P^e_{[S]}(t) - E_{[S]}(t). \tag{22}$$

Moreover, the analogous generating functions for animals of source S are:

$$J_S(t) = \frac{t^2 H'_S(t) - tU_S(t) + j(S)H_S(t) - W_S(t)}{1+t}; \tag{23}$$

$$L_S(t) = t(1+t)J_S(t); \tag{24}$$

$$P^e_S(t) = |S|H_S(t) + tH'_S(t) - J_S(t); \tag{25}$$

$$P^i_S(t) = P^e_S(t) - E_S(t), \tag{26}$$

where $j(S)$ denotes the number of pairs of adjacent pieces in the trivial heap S.

In Section 6, we give applications of this theorem to several specific lattices and sources.

4.2 Links between adjacent sites, loops and site perimeter

Our first step to prove Theorem 4.1.2 is to show that the total number of adjacent sites, loops and both site perimeters of directed animals are linked.

We begin by showing a link between adjacent sites and the site perimeters. If A is an animal, let $e(A)$ be the number of sites of A at the edge of the lattice, i.e. the number of sites with outgoing degree 1 in Γ. Thus, we have:

$$E_S(t) = \sum_A e(A)t^{|A|},$$

where the sum goes over all animals A of source S.

Lemma 4.2.1. Let A be an animal with a source included in S on Γ. The external and internal site perimeters of A verify:

$$p_e(A) = |S| + |A| - j(A);$$
$$p_i(A) = p_e(A) - e(A).$$

By summing these identities over all animals with a source included in S and of source S, we get the identities (21), (22), (25) and (26).

Proof. When dealing with the external site perimeter, the lattice Γ is embedded in a lattice Γ' which is either the full lattice or a cylindrical lattice. Let Z be the number of pairs of vertices (v, w) such that v is a site of A and w is a child of v (i.e., $v \to w$ is an edge of Γ), whether in A or not. As every vertex has outgoing degree 2, we have

$$Z = 2|A|.$$

Now, as A is a directed animal, a child of a site of A is either a site of A or a neighbour of A. The only sites and neighbours not counted are the ones in S; moreover, two sites have a child in common if and only if they are adjacent. Hence:

$$Z = |A| + p_e(A) + j(A) - |S|$$

which yields the announced formula for $p_e(A)$.

If Γ is either the half-lattice or a rectangular lattice, then each site on the edge of the lattice has one external neighbour not in Γ. Thus, we have

$$p_e(A) = p_i(A) + e(A). \qquad \square$$

The next result links the number of adjacent sites to the number of loops, and yields the identities (20) and (24).

Lemma 4.2.2. The generating functions counting the total number of adjacent sites and loops verify:

$$L_S(t) = t(1+t)J_S(t);$$
$$L_{[S]}(t) = t(1+t)J_{[S]}(t).$$

Proof. Let A be an animal with two adjacent sites marked. We regard A as a strict heap of the model $(Q, \mathcal{C})$ marked with two adjacent pieces. By pulling these two pieces downwards, we get a factorized heap $(H_1 \cdot H_2)$, with base S and such that H_1 has two maximal, adjacent pieces.

We do the same to an animal marked with a loop, and we get a factorized heap $(H_1' \cdot H_2)$ with base S such that H_1' has one maximal piece sitting on two adjacent pieces.

Placing a piece on two adjacent pieces does not alter the base or neighbourhood of H_1, so the first identity is a direct application of Lemma 2.5.6. The second easily follows by summing over all sources $T \subseteq S$. $\square$

4.3 Average number of adjacent sites

With the results of Section 4.2, all the identities of Theorem 4.1.2 are consequences of the equations (19) and (23), dealing with the average number of adjacent pieces. First, we concentrate on (19), considering animals with a source included in S. We will later derive (23) using an inclusion-exclusion argument.

To count animals marked with two adjacent sites, we regard them as heaps marked with two adjacent pieces. We then pull these two pieces downwards, creating a factorized strict heap $(H_1 \cdot H_2)$, such that H_1 has two adjacent maximal pieces.

In order to handle these factorized heaps, we consider almost-strict factorized heaps rather than strict. Define the following sets of factorized heaps:

- $\mathbb{J}^*$ the set of almost-strict factorized heaps $(H_1 \cdot H_2)$ such that H_1 has exactly two maximal pieces, which are adjacent;
- $\mathbb{M}^*$ the set of almost-strict factorized heaps $(H_1 \cdot H_2)$ such that H_1 has a unique maximal piece at a position q such that $q+2$ is in Q;
- $\mathbb{I}^{2*}$ (resp. $\mathbb{I}^{3*}$) the set of factorized heaps $(H_1 \cdot H_2)$ such that H_1 has exactly two maximal pieces, at some positions q and $q+2$ (resp. q and $q+3$).

Moreover, let $J^*_{[S]}(t)$, $M^*_{[S]}(t)$, $I^{2*}_{[S]}(t)$ and $I^{3*}_{[S]}(t)$ be the generating functions counting factorized heaps of these sets with base included in S.

Also, if $q-2$ is in S but q is not, let $W^{q*}_{[S]}(t)$ be the generating function counting almost-strict factorized heaps $(H_1 \cdot H_2)$, with base included in $S \cup \{q\}$, such that H_1 is reduced to a single piece at a position $q-2$ and H_2 has at least one minimal piece at position q. Let $W^*_{[S]}(t)$ be the sum of $W^{q*}_{[S]}(t)$ for all q in $(S+2) \setminus S$.

Lemma 2.5.5 gives the link between almost-strict and strict factorized heaps:

$$J^*_{[S]}(t) = (1+t)^2 J_{[S]}(t); \tag{27}$$

$$M^*_{[S]}(t) = (1+t)\big(tH'_{[S]}(t) - U_{[S]}(t)\big); \tag{28}$$

$$W^*_{[S]}(t) = (1+t)W_{[S]}(t). \tag{29}$$

Lemma 4.3.1. The following identity holds:

$$I^{2*}_{[S]}(t) = J^*_{[S]}(t) + tI^{3*}_{[S]}(t).$$

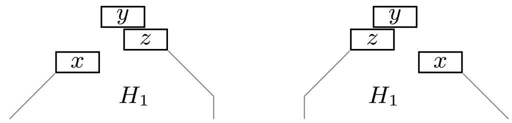
Proof. The core of this lemma is the bijection

$$\Phi : \mathbb{I}^{2*} \setminus \mathbb{J}^* \to \mathbb{I}^{3*}$$

illustrated in Figure 9. Let $(H_1 \cdot H_2)$ be a factorized heap in $\mathbb{I}^{2*} \setminus \mathbb{J}^*$. Therefore, the heap H_1 has two maximal pieces at some positions q and $q+2$, and these pieces are not at the same height. Let x be the lower of the two, and y the higher.

The heap $\Phi(H_1 \cdot H_2)$ is obtained by removing the piece y. This piece cannot be minimal, and as H_1 is strict and x is a maximal piece, y must sit on a piece z on the opposing side of x. Thus, $\Phi(H_1 \cdot H_2)$ is a heap of $\mathbb{I}^{3*}$.

Conversely, let $(H_1 \cdot H_2)$ be a factorized heap in $\mathbb{I}^{3*}$. The heap H_1 has two maximal pieces at some positions q and $q+3$. As H_1 is strict, it is aligned



$\mathbb{I}^{2*} \setminus \mathbb{J}^*$

Figure 9: The bijection Φ: after removing the piece y, a maximal piece z is uncovered at a position at a distance 3 from x.

according to Lemma 3.1.3. Therefore, the maximal pieces' heights have opposite parity and thus cannot be equal. Let x be the lower piece and z the higher: then $\Phi^{-1}(H_1 \cdot H_2)$ is obtained by adding a piece y on z at position $q+1$ or $q+2$.

In addition, let $(H_1 \cdot H_2)$ be a factorized heap of $\mathbb{I}^{2*} \setminus \mathbb{J}^*$ and let $(H_1' \cdot H_2) = \Phi(H_1 \cdot H_2)$. As the heaps H_1 and H_1' have same base and neighbourhood, Lemma 2.5.2 entails that the bijection Φ preserves the base. Moreover, it satisfies:

$$\left|\Phi(H_1 \cdot H_2)\right| = |H_1 \cdot H_2| - 1.$$

This yields the announced identity on generating functions. □

Lemma 4.3.2. The following identity holds:

$$I^{2*}_{[S]}(t) = tI^{3*}_{[S]}(t) + tM^{*}_{[S]}(t) - W^{*}_{[S]}(t).$$

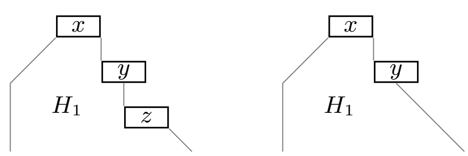
Using this lemma and the previous one, we derive (19) using (27), (28) and (29).

Proof. Again, we prove this lemma using a bijection

$$\Psi : \mathbb{I}^{2*} \to \mathbb{I}^{3*} \cup \mathbb{M}^*.$$

This bijection is illustrated in Figure 10. Let $(H_1 \cdot H_2)$ be a factorized heap of $\mathbb{I}^{2*}$. The heap H_1 has two maximal pieces x and y, at positions q and $q+2$ respectively.

Again, the factorized heap $\Psi(H_1 \cdot H_2)$ is obtained by removing the piece y. Thus, the heap $H_1 \setminus \{y\}$ may either have two or one maximal pieces. As H_1 is strict, in the former case, the new maximal piece z must be at position $q+3$, so we obtain a heap of $\mathbb{I}^{3*}$ or $\mathbb{M}^*$ respectively.



$\mathbb{I}^{3*}$ $\cup$ $\mathbb{M}^*$

Figure 10: The bijection Ψ: after removing the piece y, either a maximal piece z is uncovered at position $q+3$ or not.

The inverse mapping $\Psi^{-1}(H_1 \cdot H_2)$ is obtained by adding back a piece y on top of H_1 at position $q+2$, where q is the position of x. The bijection Ψ again

satisfies:

$$\left|\Psi(H_1 \cdot H_2)\right| = |H_1 \cdot H_2| - 1.$$

However, we cannot directly restrict the bijection Ψ to heaps of base included in S because it does not preserve the base. There are two cases when the base of $\Psi(H_1 \cdot H_2)$ is different from that of $H_1 \cdot H_2$, illustrated in Figure 11:

1. if the piece y is a minimal piece of H_1 and H_2 has no minimal piece at position $q+2$, then $q+2$ is in $b(H_1 \cdot H_2)$ but not in $b\big(\Psi(H_1 \cdot H_2)\big)$;

2. if there is a minimal piece z of H_2 with position $q+3$ and there is no piece of H_1 concurrent to it, then $q+3$ is in $b(H_1 \cdot H_2)$ but not in $b\big(\Psi^{-1}(H_1 \cdot H_2)\big)$.

Let $X^*_{[S]}(t)$ be the generating function counting the factorized heaps $(H_1 \cdot H_2)$ with base included in S and verifying condition 1 above, such that $q+2$ is not in S, and let $q' = q+2$; let $Y^*_{[S]}(t)$ be the one counting those verifying condition 2, such that $q+3$ is not in S, and let $q' = q+3$. The bijection Ψ thus entails that

$$I^{2*}_{[S]}(t) + X^*_{[S]}(t) = t\Big(I^{3*}_{[S]}(t) + M^*_{[S]}(t) + Y^*_{[S]}(t)\Big).$$

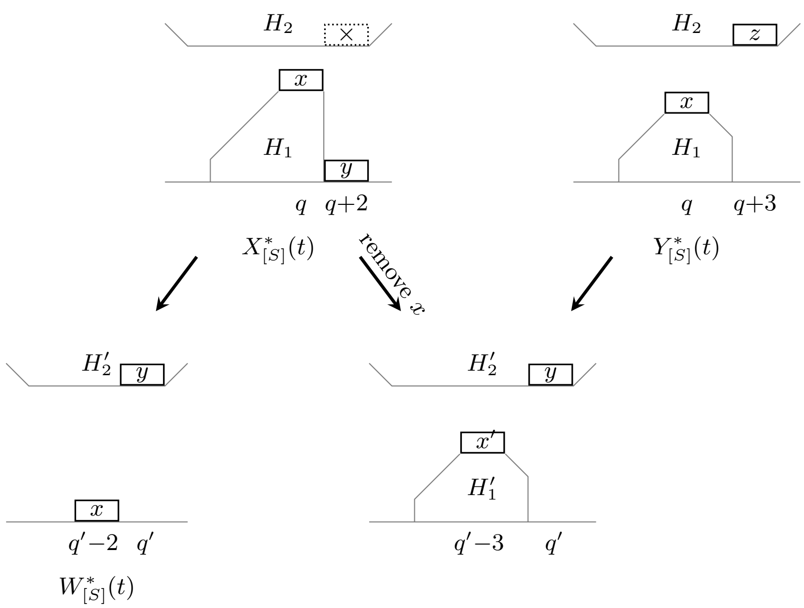


Figure 11: At the top, the two cases where the bijection Ψ does not preserve bases, counted by $X^*_{[S]}(t)$ and $Y^*_{[S]}(t)$. A heap counted by $X^*_{[S]}(t)$ is either counted by $W^*_{[S]}(t)$ or by $tY^*_{[S]}(t)$, depending on whether H_1 is reduced to the pieces x and y or not.

To simplify this equation, we perform the operations depicted in Figure 11. Let $(H_1 \cdot H_2)$ be a factorized heap counted by $X_{[S]}(t)$; let $H_1 = H_1' \cdot x \cdot y$, and $H_2' = y \cdot H_2$:

- if H_1' is empty, then the factorized heap $(x \cdot H_2')$ is counted by $W^*_{[S]}(t)$;

- if not, then as y is a minimal piece of H_1, no piece of H_1' is concurrent to q'; therefore, as H_1 is strict, H_1' has a unique maximal piece x', at position $q'-3$ (Figure 11, bottom right). Therefore, the factorized heap $(H_1' \cdot H_2')$ is counted by $Y^*_{[S]}(t)$.

Finally, we find:

$$X^*_{[S]}(t) = W^*_{[S]}(t) + tY^*_{[S]}(t).$$

This completes the proof. □

Having proved the identity (19), we now derive (23) using the inclusion-exclusion principle. This principle states that if $Z_{[S]}(t)$ and $Z_S(t)$ are generating functions verifying

$$Z_{[S]}(t) = \sum_{T \subseteq S} Z_T(t),$$

then $Z_S(t)$ is given by:

$$Z_S(t) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} Z_{[T]}(t).$$

The generating functions $H_{[S]}(t)$, $J_{[S]}(t)$ and $U_{[S]}(t)$ fit this description. To deal with $W_{[S]}(t)$, we need the lemma below.

Lemma 4.3.3. The following identity holds.

$$\sum_{T \subseteq S} (-1)^{|S \setminus T|} W_{[T]}(t) = W_S(t) - j(S) H_S(t).$$

With this lemma and the inclusion-exclusion principle applied to $J_{[S]}(t)$, $H'_{[S]}(t)$ and $U_{[S]}(t)$, the identity (23) follows from (19). We have thus finished proving Theorem 4.1.2.

Proof. Consider the generating function

$$\sum_{T \subseteq S} W_T(t).$$

Recall that $W_T(t)$ is the sum of all $W_T^q(t)$, for all q such that $q-2$ is in T but q is not. We further distinguish whether q is in S:

$$\sum_{T \subseteq S} W_T(t) = \sum_{T \subseteq S} \left(\sum_{\substack{q \notin S \\ q-2 \in T}} W_T^q(t) \right) + \sum_{T \subseteq S} \left(\sum_{\substack{q \in S \\ q-2 \in T \\ q \notin T}} W_T^q(t) \right).$$

In the first term of the r.h.s., we recognize the generating function $W_{[S]}(t)$. We rewrite the second term using the fact that $W_T^q(t) = H_{T \cup \{q\}}(t)$ and by posing $T' = T \cup \{q\}$:

$$\begin{aligned} \sum_{T \subseteq S} W_T(t) &= W_{[S]}(t) + \sum_{T' \subseteq S} \left(\sum_{\substack{q \in T' \\ q-2 \in T'}} H_{T'}(t) \right); \\ &= W_{[S]}(t) + \sum_{T' \subseteq S} j(T') H_{T'}(t). \end{aligned}$$

We rewrite this as:

$$W_{[S]}(t) = \sum_{T \subseteq S} \big(W_T(t) - j(T) H_T(t)\big).$$

Thus, we may apply the inclusion-exclusion principle on $W_{[S]}(t)$, which yields the announced formula. ☐

5 Triangular lattices

Here, we state some results on animals on the triangular lattices, similar to those of Section 4 to a certain extent. We consider the triangular lattice Δ associated to Γ, and we are interested in the average number of adjacent sites, loops and neighbours of animals of a given area.

Adjacent sites and loops are defined in Section 1. Note that the definition of loops we use is somewhat unusual; it is, for instance, different from the one found in [3]. Like on the square lattice, we now introduce several notions of site perimeter.

5.1 Site perimeter

Let A be an animal with a source included in S. We denote by $p_i(A)$ and $p_e(A)$ the internal and external site perimeters of A, defined in a manner identical to Definition 4.1.1.

Unlike on the square lattice, the generating functions giving the average value of the site perimeters are believed to be non-algebraic, and we have not been able to compute them. Instead of studying them directly, we consider the two quantities below. Let $c(A)$ be the number of sites of A only supported at the center (see Definition 3.1.1).

Definition 5.1.1. Let A be an animal with a source included in S. Define the number $p_-(A)$ to be the number of external neighbours v of A, such that v is not only supported at the center in the animal $A \cup \{v\}$. Moreover, define:

$$p_+(A) = 2|S| + |A| + c(A) - \ell(A). \tag{30}$$

Let $e(A)$ be the number of sites of A at the edge of the lattice Δ (that is, vertices having outgoing degree two).

Lemma 5.1.2. Let A be an animal with a source included in S. The external and internal site perimeters of A verify:

$$\begin{aligned} p_-(A) &= |S| + |A| + c(A) - j(A); \\ p_-(A) &\leq p_e(A) \leq p_+(A); \\ p_i(A) &= p_e(A) - e(A). \end{aligned}$$

Proof. The proof is very similar to that of Lemma 4.2.1 on the square lattice. First, we consider the external site perimeter and therefore assume that the animals live in an extended lattice Δ' where every vertex has outgoing degree three.

Let us begin with $p_-(A)$. Let Z be the number of pairs (v,w) of vertices of Δ', such that $v=(i,j)$ is a site of A and w is of the form $(i,j+1)$ or $(i+1,j)$. Obviously, we have

$$Z = 2|A|.$$

Likewise, if (v,w) is such a pair, then w is either a site of A or a neighbour of A. The only sites and neighbours not counted are the vertices of S and the sites and neighbours only supported at the center. Moreover, a vertex is counted twice every time two sites of A are adjacent. Therefore, we have:

$$Z = |A| + p_-(A) - |S| - c(A) + j(A).$$

The inequality $p_-(a) \leq p_e(A)$ is obvious by definition.

Now, let us prove the upper bound on $p_e(A)$. Let Z' be the number of pairs (v,w) of vertices of Δ', such that v is a site of A and w is a child of v. As every vertex has outgoing degree three, we have:

$$Z' = 3|A|.$$

For each such pair, w is either a site or a neighbour of A, and the only sites and neighbours not counted are the vertices of S. Moreover, as shown in Figure 12, a vertex w is counted twice at least every time a site v of A is supported on the left or on the right.

Figure 12: Each time a site v is supported on the left or on the right, a vertex w is the child of two sites.

Let v be a site of A. There are three possible cases:

- the site v is neither supported on the left nor on the right, i.e. is either in S or only supported at the center;
- the site v is supported both on the left and right, i.e. is a loop;
- the site v is either supported on the left or on the right, but not both.

Thus, we get:

$$Z' \geq |A| + p_e(A) - |S| + 2\ell(A) + \big(|A| - |S| - c(A) - \ell(A)\big)$$

which, by the definition (30), boils down to

$$p_e(A) \leq p_+(A).$$

Finally, the link between external and internal site perimeter is due to the fact that every site of A with outgoing degree two in Δ has one less neighbour in Δ than in Δ'. $\square$

5.2 Average number of adjacent sites, number of loops, and site perimeter

In the same manner as in Section 4, we define the generating function counting the total value of the parameters $j(A)$, $\ell(A)$, $p_-(A)$ and $p_+(A)$ in animals on the triangular lattice Δ. Thus, let $\mathcal{J}_S(t)$, $\mathcal{L}_S(t)$, $\mathcal{P}_S^-(t)$, $\mathcal{P}_S^+(t)$ be the generating functions dealing with animals of source S, and $\mathcal{J}_{[S]}(t)$, $\mathcal{L}_{[S]}(t)$, $\mathcal{P}_{[S]}^-(t)$ and $\mathcal{P}_{[S]}^+(t)$ be their analogues counting animals with a source included in S.

Animals on the triangular lattice Δ can be regarded as inflated heaps on the model $(Q, \mathcal{C})$ defined in Section 4. As on the square lattice, this leads us to define the following generating functions:

- $\mathcal{E}_S(t)$ counts heaps of base S marked with a piece at a position q such that either $q-1$ or $q+1$ is not in Q;
- $\mathcal{U}_S(t)$ counts heaps of base S marked with a piece at a position q such that $q+2$ is not in Q;
- if $q-2$ is in S but q is not, $\mathcal{W}_S^q(t)$ counts heaps of base $S \cup \{q\}$;
- $\mathcal{W}_S(t)$ is the sum of $\mathcal{W}_S^q(t)$ for all q in $(S+2) \setminus S$.

Finally, define the counterparts of these generating functions that count heaps of base included in S, with $\mathcal{W}_{[S]}^q(t)$ counting heaps with a base included in $S \cup \{q\}$, containing at least $q-2$ and q.

Again, the results of Section 3 enable us to compute all of these generating functions. Thus, with the theorem below, one may compute the desired generating functions.

Theorem 5.2.1. The generating functions counting the total number of adjacent sites, number of loops, and the lower and upper bound on the site perimeter in animals with a source included in S are:

$$\mathcal{J}_{[S]}(t) = \frac{t^2\mathcal{H}'_{[S]}(t) - t\mathcal{U}_{[S]}(t) - \mathcal{W}_{[S]}(t)}{1+t}; \tag{31}$$

$$\mathcal{L}_{[S]}(t) = t\mathcal{J}_{[S]}(t); \tag{32}$$

$$\mathcal{P}_{[S]}^-(t) = |S|\mathcal{H}_{[S]}(t) + (t+t^2)\mathcal{H}'_{[S]}(t) - \mathcal{J}_{[S]}(t); \tag{33}$$

$$\mathcal{P}_{[S]}^+(t) = 2|S|\mathcal{H}_{[S]}(t) + (t+t^2)\mathcal{H}'_{[S]}(t) - \mathcal{L}_{[S]}(t). \tag{34}$$

The generating functions counting animals with source S are:

$$\mathcal{J}_S(t) = \frac{t^2\mathcal{H}'_S(t) - t\mathcal{U}_S(t) + j(S)\mathcal{H}_S(t) - \mathcal{W}_S(t)}{1+t}; \tag{35}$$

$$\mathcal{L}_S(t) = t\mathcal{J}_S(t); \tag{36}$$

$$\mathcal{P}_S^-(t) = |S|\mathcal{H}_S(t) + (t+t^2)\mathcal{H}'_S(t) - \mathcal{J}_S(t); \tag{37}$$

$$\mathcal{P}_S^+(t) = 2|S|\mathcal{H}_S(t) + (t+t^2)\mathcal{H}'_S(t) - \mathcal{L}_S(t). \tag{38}$$

First, let us show the identities (33), (34), (37), (38). Let $\mathcal{C}_{[S]}(t)$ (resp. $\mathcal{C}_S(t)$) be the generating functions counting the total number of sites only supported at

the center in animals with a source included in S (resp. of source S). Equation (12), found in Section 2, implies that:

$$\mathcal{C}_{[S]}(t) = t^2\mathcal{H}'_{[S]}(t);$$
$$\mathcal{C}_S(t) = t^2\mathcal{H}'_S(t).$$

Therefore, equations (33), (34), (37) and (38) are found by summing the first identity of Lemma 5.1.2 and idendity (30) on all the relevant animals.

Moreover, the external and internal site perimeters, like on the square lattice, are linked by:

$$\mathcal{P}^i_S(t) = \mathcal{P}^e_S(t) - \mathcal{E}_S(t).$$

The case of animals with a source included in S is identical.

The following lemma proves identities (32) and (36).

Lemma 5.2.2. The generating functions counting the total number of adjacent sites and loops are linked by:

$$\mathcal{L}_{[S]}(t) = t\mathcal{J}_{[S]}(t);$$
$$\mathcal{L}_S(t) = t\mathcal{J}_S(t).$$

Proof. The proof is also similar to that of Lemma 4.2.2 for the square lattice. Let A be an animal of source S with two adjacent sites marked. We regard A as an inflated heap of the model $(Q, \mathcal{C})$ marked with two adjacent pieces, and turn it into a factorized heap $(H_1 \cdot H_2)$ by pulling the marked pieces downwards, such that H_1 has two maximal, adjacent pieces.

We do the same with animals marked with a loop. The resulting factorized heaps differ only by a piece on top of H_1, which does not alter the base or neighbourhood of H_1. Therefore, Lemma 2.5.3 provides the desired link on the generating functions. □

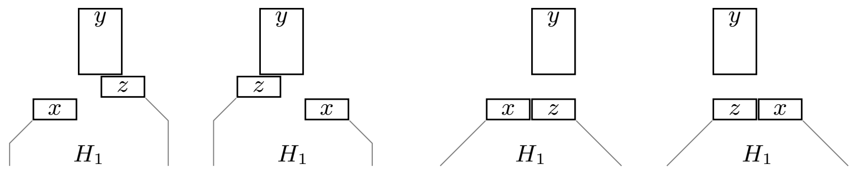
To deal with the number of adjacent sites, we use the same intermediate as for the square lattice. Let $\mathcal{I}^2_{[S]}(t)$ (resp. $\mathcal{I}^3_{[S]}$) be the generating function counting factorized heaps $(H_1 \cdot H_2)$, such that H_1 has two maximal pieces, with positions q and $q+2$ (resp. q and $q+3$) for some q. These functions are linked to $\mathcal{J}_{[S]}(t)$ in the following manner.

Lemma 5.2.3. The following identity holds:

$$\mathcal{J}_{[S]}(t) = \frac{1-t}{1+t}\mathcal{I}^2_{[S]}(t) - \frac{t}{1+t}\mathcal{I}^3_{[S]}(t).$$

Proof. As the generating functions counting heaps and inflated heaps are the same, we may regard the factorized heaps counted by $\mathcal{I}^2_{[S]}(t)$ and $\mathcal{I}^3_{[S]}(t)$ as factorized inflated heaps, i.e. factorized pre-heaps $(H_1 \cdot H_2)$ where H_1 and H_2 are inflated heaps.

Let $(H_1 \cdot H_2)$ be an inflated factorized heap with base included in S such that H_1 has two maximal pieces x and y, which are at positions at distance 2 but are not adjacent (say, y is higher than x). We transform it by removing from H_1 all pieces of the stack of y that are higher than x. The resulting factorized heap $(H'_1 \cdot H_2)$ is such that H'_1 has two maximal pieces, either adjacent or at positions at distance 3 (see Figure 13).



$$\frac{t}{1-t}\mathcal{I}^3_{[S]}(t) \qquad + \qquad \frac{t}{1-t}\mathcal{J}_{[S]}(t) \quad + \quad \frac{t}{1-t}\mathcal{J}_{[S]}(t)$$

Figure 13: Removing the pieces of the stack of y that are higher than x uncovers a maximal piece z, either at a position at distance 3 from x or adjacent to x.

We now consider the inverse transformation. Let $(H_1 \cdot H_2)$ be an inflated heap with base included in S such that H_1 has two maximal pieces at positions at distance 3. As S is aligned, H_1 is also aligned, so its maximal pieces cannot be at the same height: let z be the higher piece and x the lower. The inverse transformation is performed by adding a stack of arbitrary height on top of z, at a distance 2 from x.

Finally, let $(H_1 \cdot H_2)$ be a inflated heap of base included in S, such that H_1 has two maximal, adjacent pieces. This time, the inverse transformation is performed by adding a stack of arbitrary height on top of either maximal piece, so that these heaps must be counted twice. We thus have:

$$\mathcal{I}^2_{[S]}(t) - \mathcal{J}_{[S]}(t) = \frac{t}{1-t}\big(\mathcal{I}^3_{[S]}(t) + 2\mathcal{J}_{[S]}(t)\big),$$

which yields the announced formula for $\mathcal{J}_{[S]}(t)$. □

We complete the proof of Theorem 5.2.1 by using the following result in conjunction with the results of Section 4.

Lemma 5.2.4. The following links hold:

$$\begin{aligned}
\mathcal{I}^2_{[S]}(t) &= I^{2*}_{[S]}(t)\left(\frac{t}{1-t}\right); \\
\mathcal{I}^3_{[S]}(t) &= I^{3*}_{[S]}(t)\left(\frac{t}{1-t}\right); \\
t\mathcal{H}'_{[S]}(t) - \mathcal{U}_{[S]}(t) &= M^*_{[S]}\left(\frac{t}{1-t}\right); \\
\mathcal{W}_{[S]}(t) &= W_{[S]}\left(\frac{t}{1-t}\right).
\end{aligned}$$

Proof. The fourth identity is an instance of (9) (Section 2.2): it is due to the fact that every heap is obtained by replacing each piece of a strict heap by a stack of pieces.

To prove the first three identities, we regard the heaps counted by $\mathcal{U}_{[S]}(t)$ as factorized heaps by pulling the marked piece downwards. In this way, these identities link generating functions counting factorized heaps and almost-stict factorized heaps. This link is found by replacing each piece of both parts of an almost-strict factorized heap by a stack of pieces. □

Thus, combining identity (29), Lemmas 4.3.2 and 4.3.3 and this result, we find:

$$\mathcal{I}^2_{[S]}(t) = \frac{t}{1-t}\Big(\mathcal{I}^3_{[S]}(t) + t\mathcal{H}'_{[S]}(t) - \mathcal{U}_{[S]}(t) - \mathcal{W}_{[S]}(t)\Big);$$
$$\sum_{T\subseteq S}(-1)^{|S\setminus T|}\mathcal{W}_{[T]}(t) = \mathcal{W}_S(t) - j(S)\mathcal{H}_S(t).$$

Lemma 5.2.3 and an inclusion-exclusion argument yield (31) and (35).

6 Applications

6.1 Asymptotic results

We now use Theorems 4.1.2 and 5.2.1 to compute asymptotic estimates of the average value of the various parameters in animals of a given area. Let S be a one-line source, and let μ be the growth constant of the animals of source S on the square lattice Γ. Propositions 3.2.4 and 3.3.6 assert that this constant does not depend on S, and that the growth constant of animals of source S on the associated triangular lattice Δ is $\bar\mu = \mu + 1$.

Moreover, let $j(n)$, $\ell(n)$, $p_i(n)$ and $p_e(n)$ be the average number of adjacent sites, number of loops, internal and external site perimeter of animals of source S and area n on Γ; let $\bar\jmath(n)$, $\bar\ell(n)$, $\bar p_e(n)$, $\bar p_i(n)$, $\bar p_-(n)$ and $\bar p_+(n)$ be the average value of each parameter of the animals on Δ.

Corollary 6.1.1. Assume that Γ is either the full lattice, the half lattice, or a cylindrical bounded lattice. As n tends to infinity, we have the following estimates:

$$j(n) \sim \frac{n}{\mu+1}; \qquad \bar\jmath(n) \sim \frac{n}{\bar\mu+1};$$
$$\ell(n) \sim \frac{n}{\mu^2}; \qquad \bar\ell(n) \sim \frac{n}{\bar\mu(\bar\mu+1)};$$
$$p_i(n) \sim p_e(n) \sim \frac{\mu}{\mu+1}n; \qquad \bar p_-(n) \sim \left(1 + \frac{1}{\bar\mu(\bar\mu+1)}\right)n;$$
$$\bar p_+(n) \sim \left(1 + \frac{1}{\bar\mu+1}\right)n.$$

Moreover, the average site perimeters on the triangular lattice verify for large n:

$$\bar p_-(n) \le \bar p_i(n) \sim \bar p_e(n) \le \bar p_+(n).$$

In the unbounded lattices, the growth constants are $\mu = 3$ and $\bar\mu = 4$. Thus, these estimates become:

$$j(n) \sim \frac{n}{4}; \qquad p(n) \sim \frac{3n}{4}; \qquad \bar\jmath(n) \sim \frac{n}{5}; \qquad \bar p_-(n) \sim \frac{21n}{20};$$
$$\ell(n) \sim \frac{n}{9}; \qquad \bar\ell(n) \sim \frac{n}{20}; \qquad \bar p_+(n) \sim \frac{6n}{5}.$$

Proof. We use singularity analysis on the identities of Theorems 4.1.2 and 5.2.1. Let us begin with the square lattice Γ. Let $(Q, \mathcal{C})$ be the corresponding model of heaps of dominoes. As Γ is not a rectangular lattice, the position $q+2$ is in Q as soon as q is. Thus, the generating function $U_S(t)$ is zero.

According to Propositions 3.2.4 and 3.3.6, the asymptotic behaviour of the coefficients of $H_S(t)$ and $W_S(t)$ differ only by a multiplicative constant; thus, the only meaningful term in $J_S(t)$ is $\frac{t^2}{1+t}H'_S(t)$. This term yields the announced estimate of $\bar{\jmath}(n)$, and the other parameters follow.

All that is left to prove is that the term $E_S(t)$, needed to compute the total internal site perimeter, is negligible. If Γ is the full lattice or a cylindrical lattice, this term is zero. If Γ is the half-lattice, it counts heaps of base S marked with a piece at position 0:

$$E_S(t) = H_S^{(0)}(t).$$

According to Proposition 2.6.3, we have:

$$E_S(t) \leq \frac{1}{1+t} H_{[S]}(t) H_{\{0\}}(t),$$

where the inequality holds coefficient-by-coefficient. Using Proposition 3.3.6, we see that this is equivalent to $\lambda\sqrt{1-3t}$ as t tends to $1/3$, which yields negligible coefficients in comparison to $H'_S(t)$.

The case of the triangular lattice is similar. □

In this corollary, the rectangular lattices are not included. Computing the estimates in this case is more difficult, because these lattices are the only ones where the generating functions $U_S(t)$ and $E_S(t)$ have non-negligible coefficients compared to those of $H'_S(t)$.

6.2 Examples

6.2.1 Single-source animals on the full lattices

In all the cases covered by the results of Sections 4 and 5, the simplest is the one of single-source directed animals on the square lattice.

Corollary 6.2.1. The generating functions counting the total number of adjacent sites, number of loops and site perimeter of the single-source directed animals on the full square lattice are respectively given by:

$$\begin{aligned}
J(t) &= \frac{1}{2t(1+t)}\left(1 - \frac{1-4t+t^2+4t^3}{\sqrt{1+t}(1-3t)^{3/2}}\right);\\
L(t) &= \frac{1}{2}\left(1 - \frac{1-4t+t^2+4t^3}{\sqrt{1+t}(1-3t)^{3/2}}\right);\\
P(t) &= \frac{1}{2t(1+t)}\left(-1+t+t^2 + \frac{1-3t+2t^2+t^3-3t^4}{\sqrt{1+t}(1-3t)^{3/2}}\right).
\end{aligned}$$

The value of $P(t)$ was conjectured by Conway [5], and the value of $L(t)$ was proved by Bousquet-Mélou using a gas model method [3].

Proof. To compute $J(t)$, we use Theorem 4.1.2 with $S = \{0\}$. The generating function $U_{\{0\}}(t)$ is zero; $W_{\{0\}}(t)$ counts strict heaps of base $\{0, 2\}$, that is,

animals with a compact source with two sites. The quantity $j(\{0\})$ is zero. Lemma 3.3.3 thus entails that:

$$J(t) = \frac{t^2 A'(t) - D(t)A(t)}{1+t}$$

where the generating functions $A(t)$ and $D(t)$ are given by (17) and (15). The values of $L(t)$ and $P(t)$ follow from Theorem 4.1.2. □

Corollary 6.2.2. The generating functions counting the total number of adjacent sites, numer of loops, and bounds on the site perimeter of single-source animals on the full triangular lattice are:

$$\begin{aligned}
\mathcal{J}(t) &= \frac{1}{2t(1+t)}\left(1 - t - \frac{1-7t+12t^2-2t^3}{(1-4t)^{3/2}}\right);\\
\mathcal{L}(t) &= \frac{1}{2(1+t)}\left(1 - t - \frac{1-7t+12t^2-2t^3}{(1-4t)^{3/2}}\right);\\
\mathcal{P}^-(t) &= \frac{1}{2t(1+t)}\left(-1 - t^2 + \frac{1-6t+11t^2-52t^3+2t^4}{(1-4t)^{3/2}}\right);\\
\mathcal{P}^+(t) &= \frac{1}{2(1+t)}\left(-3 - t + \frac{3-11t+8t^2}{(1-4t)^{3/2}}\right).
\end{aligned}$$

This time, the value of $\mathcal{L}(t)$ is different from the one found by Bousquet-Mélou, who used a different definition of loops.

Proof. This time, we use Theorem 5.2.1. Exactly as above, we have:

$$\mathcal{J}(t) = \frac{t^2 \mathcal{A}'(t) - \mathcal{D}(t)\mathcal{A}(t)}{1+t}$$

where $\mathcal{A}(t)$ and $\mathcal{D}(t)$ are given by (18) and (16). Once again, Theorem 5.2.1 enables us to derive the three other generating functions. □

6.2.2 Compact-source animals on the full lattices

According to Proposition 3.3.4, the number of compact-source animals on both the square and triangular lattice is extremely simple. Below are the generating functions counting the total number of adjacent sites of these animals.

Corollary 6.2.3. The generating functions counting the total number of adjacent sites of the compact-source directed animals on the full square and triangular lattices are given by:

$$\begin{aligned}
J_C(t) &= \frac{1}{2}\left(\frac{1-2t}{\sqrt{1+t}(1-3t)^{3/2}} - \frac{1-3t-2t^2}{(1+t)(1-3t)^2}\right).\\
\mathcal{J}_C(t) &= \frac{1}{2(1+t)}\left(\frac{1-3t}{(1-4t)^{3/2}} - \frac{1-5t+2t^2}{(1-4t)^2}\right).
\end{aligned}$$

Of course, although we only give the generating functions for the total number of adjacent pieces, Theorems 4.1.2 and 5.2.1 can still be used to deal with the total number of loops and total site perimeter.

Proof. Let $k \geq 1$; we use Theorem 4.1.2 on animals on the square lattice of source C_k (the compact source with k sites). The generating function $U_{C_k}(t)$ is zero. We remark that W_{C_k} counts animals of source C_{k+1}; moreover, there are $k-1$ pairs of adjacent sites in C_k. Thus:

$$J_{C_k}(t) = \frac{t^2 H'_{C_k}(t) + (k-1)H_{C_k}(t) - H_{C_{k+1}}(t)}{1+t}.$$

On the triangular lattice, Theorem 5.2.1 entails that:

$$\mathcal{J}_{C_k}(t) = \frac{t^2 \mathcal{H}'_{C_k}(t) + (k-1)\mathcal{H}_{C_k}(t) - \mathcal{H}_{C_{k+1}}(t)}{1+t}.$$

Lemma 3.3.3 gives the value of the generating functions $H_{C_k}(t)$ and $\mathcal{H}_{C_k}(t)$. Summing these identities for all $k \geq 1$ and using the formula

$$\sum_{k \geq 1} kD(t)^{k-1} = \frac{1}{\big(1-D(t)\big)^2},$$

we obtain:

$$J_C(t) = \frac{1}{1+t}\left(t^2 A'_C(t) + \frac{A(t)}{\big(1-D(t)\big)^2} - A_C(t) - \big(A_C(t) - A(t)\big)\right).$$

The generating function $\mathcal{J}_C(t)$ is computed in the same manner. Recall that the generating functions counting compact-source animals are:

$$A_C(t) = \frac{t}{1-3t}; \qquad \mathcal{A}_C(t) = \frac{t}{1-4t}. \qquad \square$$

6.2.3 Single-source half-animals on the square rectangular lattices and half-lattice

Let $m \geq 1$. We consider half-animals on the bounded rectangular lattice of width m (or animals with a single source at the corner of the lattice). Our goal is to compute the generating functions giving the average site perimeters of these half-animals, both internal and external.

Corollary 6.2.4. The generating functions counting the total external and internal site perimeter of half-animals in the rectangular square lattice of width m are:

$$P^e_m(t) = D_m(t) + \frac{t}{1+t}D'_m(t) + \frac{1}{1+t}D_m(t)^2;$$
$$P^i_m(t) = \frac{t}{1+t}D_m(t) + \frac{t}{1+t}D'_m(t) - \frac{1}{1+t}\big(D_m(t) - D_{m-2}(t)\big)D_m(t),$$

where $D_m(t)$ is given in terms of the polynomials computed by Proposition 3.2.1:

$$D_m(t) = \frac{T_{m-1}\big(\frac{t}{1+t}\big)}{T_m\big(\frac{t}{1+t}\big)} - 1.$$

Proof. The model of heaps of dominoes corresponding to the rectangular lattice of width m is $Q = [0, m-1]$. Instead of studying pyramids of base 0 in this model, we choose, by symmetry, to study pyramids of basis $m-1$. These pyramids are still counted by $D_m(t)$, according to Proposition 3.2.2.

We now compute all the necessary generating functions to apply Theorem 4.1.2 with $S = \{m-1\}$. As $m+1$ is not in Q, the generating function $W_{\{m-1\}}$ is zero; the quantity $j(\{m-1\})$ is also zero. The two remaining generating functions are given by:

$$U_{\{m-1\}}(t) = H^{(m-1)}_{\{m-1\}}(t) + H^{(m-2)}_{\{m-1\}}(t);$$
$$E_{\{m-1\}}(t) = H^{(m-1)}_{\{m-1\}}(t) + H^{(0)}_{\{m-1\}}(t),$$

where $H^{(q)}_{\{m-1\}}(t)$ counts strict heaps of base $m-1$ marked with a piece at position q.

As no heap marked with a piece can have an empty base, the generating functions $H^{q}_{\{m-1\}}(t)$ and $H^{q}_{[\{m-1\}]}(t)$ are equal. We compute them using Proposition 2.6.3; for this, we need the generating functions counting strict heaps of base included in $\{m-1\}$ avoiding the positions $m-2$ and 0. As no nonempty pyramid of base $m-1$ can avoid $m-2$, we have:

$$V^{m-2}_{[\{m-1\}]}(t) = 1.$$

Moreover, the pyramids of base $m-1$ avoiding 0 are the pyramids of the model $Q = [2, m-1]$, which is equivalent to the rectangular model with $m-2$ positions. Thus, we get:

$$V^{0}_{[\{m-1\}]}(t) = 1 + D_{m-2}(t).$$

The generating functions $H_{\{m-1\}}(t)$ and $H_{\{0\}}(t)$ are both $D_m(t)$. Finally, to compute $H_{\{m-2\}}(t)$, we remark that a strict pyramid of base $m-1$ is either reduced to a single piece at position $m-1$, or is a piece at position $m-1$ surmounted by a strict pyramid of base $m-2$. This entails that:

$$H_{\{m-1\}}(t) = t\big(1 + H_{\{m-2\}}(t)\big).$$

Hence:

$$H_{\{m-2\}}(t) = \frac{D_m(t)}{t} - 1.$$

Putting this all together, we get, using Proposition 2.6.3:

$$U_{\{m-1\}}(t) = \frac{1}{1+t}\left[\big(1 + D_m(t)\big)D_m(t) + D_m(t)\left(\frac{D_m(t)}{t} - 1\right)\right];$$
$$E_{\{m-1\}}(t) = \frac{1}{1+t}\Big[\big(1 + D_m(t)\big)D_m(t) + \big(D_m(t) - D_{m-2}(t)\big)D_m(t)\Big].$$

The first identity simplifies into:

$$U_{\{m-1\}}(t) = \frac{D_m(t)^2}{t}.$$

We conclude using Theorem 4.1.2. □

The value of the generating functions $P^e_m(t)$ were conjectured by Le Borgne [10], who gave an induction formula to compute them. By using the formula above, one can prove this conjecture. Finally, by simply letting m tend to infinity, this result extends to the unbounded half-lattice.

Corollary 6.2.5. The generating functions counting the total external and internal site perimeters of half-animals on the unbounded lattice are:

$$\begin{aligned} P^e(t) &= D(t) + \frac{t}{1+t} D'(t) + \frac{1}{1+t} D(t)^2; \\ P^i(t) &= \frac{t}{1+t} \big(D(t) + D'(t) \big). \end{aligned}$$

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